The relation between the factor systems of a group and those of an invariant subgroup

P M van den Broek

Institute for Theoretical Physics, University of Nymegen, the Netherlands

Abstract. The relation between the factor systems of a group and the factor systems of an invariant subgroup is discussed both for PU and for PUA representations. The results are used to discuss the factor systems of a class of magnetic space groups.

1. Introduction

From the well-known results of Wigner and Bargmann it follows that the study of projective unitary (PU) and projective unitary–anti-unitary (PUA) representations of symmetry groups of physical systems is of great importance in physics. A natural consequence of dealing with PU and PUA representations is the study of their factor systems.

In this paper we will give a decomposition of the factor systems of a group which possesses an invariant subgroup. Such a decomposition is useful for the determination of a complete set of inequivalent factor systems, as we will show in §§ 4 and 5. For factor systems of PU representations, decompositions have been given previously for two special cases: by Mackey (1958, see pp 303–4 especially) for the case where the group is a semi-direct product and by Backhouse and Bradley (1972) for the case where the invariant subgroup is of prime order. Mackey's result will turn out to be a special case of our result. Bradley and Wallis (1974) gave a decomposition for the factor systems of PUA representations for the case where the subgroup is of index two and is represented by unitary operators. Both the results of Backhouse and Bradley and those of Bradley and Wallis can easily be derived from our results.

The plan of the paper is as follows: in § 2 we give some mathematical preliminaries to fix the notation, the main theorems are presented in § 3, and in §§ 4 and 5 we apply the results to those magnetic space groups which are semi-direct products of a black and white lattice and a point group with unitary elements.

2. Preliminaries

Let G be a group and G_0 a subgroup of G of index 1 or 2. A PUA representation of G with respect to G_0 is a mapping D from G into the operators on some Hilbert space $\mathcal{H}$ such that:

(i) the operator $D(g)$ is unitary if $g \in G_0$ and anti-unitary if $g \notin G_0$.

(ii) $D(g)D(g') = \sigma(g, g')D(gg')$ for all $g, g' \in G$ for some mapping $\sigma : G \times G \to U(1)$.

(iii) $D(e) = I$, where e is the identity of G and I is the identity operator on $\mathcal{H}$. The mapping σ satisfies

$$\sigma(g, e) = \sigma(e, g) = 1 \qquad \forall g \in G \tag{2.1}$$

and

$$\sigma(g, g')\sigma(gg', g'') = \sigma(g, g'g'')\sigma^g(g', g'') \tag{2.2}$$

where λ^g is defined by

$$\lambda^g = \begin{cases} \lambda & \text{if } g \in G_0 \\ \lambda^* & \text{if } g \notin G_0 \end{cases} \tag{2.3}$$

and the asterisk denotes complex conjugation.

A mapping $\sigma : G \times G \to U(1)$ which satisfies equations (2.1) and (2.2) is called a factor system of G with respect to G_0. In the following a factor system of G shall always mean a factor system of G with respect to G_0. If D is a PUA representation of G with factor system σ and c is a mapping from G into $U(1)$ with $c(e) = 1$ then $D'(g) = c(g)D(g)$ is a PUA representation of G with factor system

$$\sigma'(g_1, g_2) = [c(g_1)c^{g_1}(g_2)/c(g_1g_2)]\sigma(g_1, g_2). \tag{2.4}$$

Two factor systems σ and σ' of G are called equivalent if a mapping $c : G \to U(1)$ with $c(e) = 1$ exists such that equation (2.4) holds. If G_0 is a subgroup of index 1, i.e. $G = G_0$ then D is a PU representation.

Now suppose H is an invariant subgroup of G and K is the quotient group G/H. The elements of K are the cosets of G with respect to H. Elements of H will be denoted by $a, b, c, \ldots$, and elements of K by $\alpha, \beta, \gamma, \ldots$. Take for each coset $\alpha \in K$ a representative $r(\alpha)$, and let $r(\epsilon) = e$, if ϵ is the unit element of K. Then each element of G can uniquely be written as (a, α) if we define

$$(a, \alpha) = ar(\alpha). \tag{2.5}$$

Let a^α be defined by

$$a^\alpha = r(\alpha)ar^{-1}(\alpha) \tag{2.6}$$

and the mapping $m : K \times K \to H$ by

$$r(\alpha)r(\beta) = m(\alpha, \beta)r(\alpha\beta). \tag{2.7}$$

From the associativity, it then follows

$$m(\alpha, \beta)m(\alpha\beta, \gamma) = m(\alpha, \beta\gamma)m^\alpha(\beta, \gamma) \tag{2.8}$$

and we also have

$$(a^\alpha)^\beta = m(\beta, \alpha)a^{\beta\alpha}m^{-1}(\beta, \alpha). \tag{2.9}$$

The multiplication of the elements of G is given by

$$(a, \alpha)(b, \beta) = (ab^\alpha m(\alpha, \beta), \alpha\beta).$$

In the following we shall write λ^a and λ^α for brevity if we mean $\lambda^{(a,\epsilon)}$ and $\lambda^{(e,\alpha)}$ respectively.

For the case where G_0 is a subgroup of index 2 we have to distinguish two possibilities:

(i) $H \subseteq G_0$. In this case K has a subgroup K_0 of index 2 such that $(a, \alpha) \in G_0$ if and only if $\alpha \in K_0$.

(ii) $H \nsubseteq G_0$. Now H has a subgroup H_0 of index 2 which equals $H \cap G_0$. Further, each coset of G with respect to H contains both elements of G_0 and elements of $G \backslash G_0$. Therefore we may choose the coset representatives $r(\alpha)$ to belong to G_0.

In the following we assume such choice has been made. Then $m(\alpha, \beta) \in H_0$ for all $\alpha, \beta \in K$ and $(a, \alpha) \in G_0$ if and only if $a \in H$.

3. The main theorems

Theorem 1.
Let ω' be a factor system of G. There exists an equivalent factor system ω of G which decomposes as follows:

$$\omega((a, \alpha), (b, \beta)) = \gamma(a, b^\alpha)\gamma(ab^\alpha, m(\alpha, \beta))\nu^{ab}(\alpha, \beta)P^a(\alpha, b) \qquad (3.1)$$

where γ is a factor system of H with respect to H_0, ν is a mapping from $K \times K$ into $U(1)$ and P is a mapping from $K \times H$ into $U(1)$ with the following properties:

$$\nu(\epsilon, \alpha) = \nu(\alpha, \epsilon) = 1 \qquad \forall \alpha \in K \qquad (3.2)$$

$$P(\epsilon, a) = P(\alpha, e) = 1 \qquad \forall \alpha \in K, \forall a \in H \qquad (3.3)$$

$$P(\alpha, bc) = P(\alpha, b)P^b(\alpha, c)\frac{\gamma(b^\alpha, c^\alpha)}{\gamma^\alpha(b, c)} \qquad (3.4)$$

$$P(\alpha\beta, c) = P(\alpha, c^\beta)P^\alpha(\beta, c)\frac{\nu^c(\alpha, \beta)}{\nu(\alpha, \beta)} \frac{\gamma(m(\alpha, \beta)c^{\alpha\beta}m^{-1}(\alpha, \beta), m(\alpha, \beta))}{\gamma(m(\alpha, \beta), c^{\alpha\beta})} \qquad (3.5)$$

$$\frac{\nu(\alpha, \beta)\nu(\alpha\beta, \gamma)}{\nu(\alpha, \beta\gamma)\nu^\alpha(\beta, \gamma)} = \frac{\gamma(m^\alpha(\beta, \gamma), m(\alpha, \beta\gamma))}{\gamma(m(\alpha, \beta), m(\alpha\beta, \gamma))}P(\alpha, m(\beta, \gamma)). \qquad (3.6)$$

Proof.
Define the mapping $c : G \rightarrow U(1)$ by $c(a, \alpha) = \omega'((a, \epsilon), (e, \alpha))$ and the factor system ω of G by $\omega(g, g') = [c(g)c^g(g')/c(gg')]\omega'(g, g')$. Then $\omega((a, \epsilon), (e, \alpha)) = 1$. Let D be a PUA representation of G with factor system ω. Then $D((a, \epsilon))D((e, \alpha)) = D((a, \alpha))$. Now

$$\omega((a, \alpha), (b, \beta))D((ab^\alpha m(\alpha, \beta), \alpha\beta))$$

$$= D((a, \alpha))D((b, \beta)) = D((a, \epsilon))D((e, \alpha))D((b, \epsilon))D((e, \beta)).$$

On the other hand we have

$$\omega((a, \epsilon), (b^\alpha, \epsilon))\omega((ab^\alpha, \epsilon), (m(\alpha, \beta), \epsilon))\omega^{ab}((e, \alpha), (e, \beta))D((ab^\alpha m(\alpha, \beta), \alpha\beta))$$

$$= \omega((a, \epsilon), (b^\alpha, \epsilon))\omega^{ab}((e, \alpha), (e, \beta))D((ab^\alpha, \epsilon))D((m(\alpha, \beta), \epsilon))D((e, \alpha\beta))$$

$$= \omega((a, \epsilon), (b^\alpha, \epsilon))D((ab^\alpha, \epsilon))\omega((e, \alpha), (e, \beta))D((m(\alpha, \beta), \alpha\beta))$$

$$= D((a, \epsilon))D((b^\alpha, \epsilon))D((e, \alpha))D((e, \beta)).$$

If we multiply these two relations on the left with $D^{-1}((a, \epsilon))$ and on the right with $D^{-1}((e, \beta))$ and combine them we obtain

$$D((e, \alpha))D((b, \epsilon))$$

$$= \frac{\omega^{a}((a, \alpha), (b, \beta))}{\omega^{a}((a, \epsilon), (b^{\alpha}, \epsilon))\omega^{a}((ab^{\alpha}, \epsilon), (m(\alpha, \beta), \epsilon))\omega^{b}((e, \alpha), (e, \beta))}$$

$$D((b^{\alpha}, \epsilon))D((e, \alpha)).$$

Since the numerical factor on the right-hand side depends on α and b only we may denote it by $P(\alpha, b)$.

Let the mapping $\nu : K \times K \to U(1)$ be defined by $\nu(\alpha, \beta) = \omega((e, \alpha), (e, \beta))$ and let γ be the restriction of ω to $H \times H$. Then γ is a factor system of H, and equation (3.1) has been proved. Equation (3.2) follows from the definition of ν and equation (3.3) follows from equation (3.1) if we take $\alpha = \epsilon$; $a = e$ and $b = e$; $\beta = \epsilon$ respectively. Now from the definition of a factor system we have

$$\omega((a, \alpha)(b, \beta))\omega((ab^{\alpha}m(\alpha, \beta), \alpha\beta), (c, \gamma))$$

$$= \omega((a, \alpha), (bc^{\beta}m(\beta, \gamma), \beta\gamma))\omega^{(a,\alpha)}((b, \beta), (c, \gamma)).$$

If we take $\beta = \epsilon$ in this equation and use equation (3.1), the result is equation (3.4). In the same way equation (3.5) follows if we take $a = b = e$ and $\gamma = \epsilon$. Finally equation (3.6) follows if we take $a = b = c = e$. This proves the theorem.

The opposite of the theorem above is also true:

Theorem 2.
Let γ be a factor system of H with respect to H_0. Let ν and P be mappings from $K \times K$ and $K \times H$ respectively into $U(1)$ such that the equations (3.2), (3.3), (3.4), (3.5) and (3.6) are satisfied. Then the mapping $\omega : G \times G \to U(1)$ defined by equation (3.1) is a factor system of G.

The proof of this theorem is a straightforward calculation and is therefore omitted.

The question when two factor systems decomposed as in equation (3.1) are equivalent is answered by the following theorem.

Theorem 3.
A factor system ω decomposed as in equation (3.1) is equivalent with the trivial factor system if and only if there exist mappings $d : H \to U(1)$ and $e : K \to U(1)$ with $d(e) = e(\epsilon) = 1$ such that γ, P and ν can be written as

$$\nu(\alpha, \beta) = \frac{e(\alpha)e^{\alpha}(\beta)}{d(m(\alpha, \beta))e(\alpha\beta)} \tag{3.7}$$

$$P(\alpha, b) = \frac{e(\alpha)d^{\alpha}(b)}{d(b^{\alpha})e^{b}(\alpha)} \tag{3.8}$$

and

$$\gamma(a, b) = \frac{d(a)d^{a}(b)}{d(ab)}. \tag{3.9}$$

Proof.

If ω is equivalent with the trivial factor system then there exists a mapping $c: G \to U(1)$, with $c((e, \epsilon)) = 1$ such that

$$\gamma(a, b^{\alpha})\gamma(ab^{\alpha}, m(\alpha, \beta))v^{ab}(\alpha, \beta)P^{a}(\alpha, b) = \frac{c((a, \alpha))c^{(a,\alpha)}((b, \beta))}{c((ab^{\alpha}m(\alpha, \beta), \alpha\beta))}. \tag{3.10}$$

If we define d and e by $d(a) = c((a, \epsilon))$ and $e(\alpha) = c((e, \alpha))$ then it follows from equation (3.10) with $b = e$ and $\alpha = \epsilon$

$$c((a, \beta)) = d(a)e^{a}(\beta). \tag{3.11}$$

Now the equations (3.7), (3.8) and (3.9) follow immediately from (3.10) and (3.11) by taking the appropriate elements equal to the unit element.

On the other hand, if equations (3.7), (3.8) and (3.9) hold for some mappings $d: H \to U(1)$ and $e: K \to U(1)$ with $d(e) = e(\epsilon) = 1$ and the mapping $c: G \to U(1)$ is defined by equation (3.11) then it is easily verified that equation (3.10) holds, which means that ω is equivalent with the trivial factor system.

4. The factor systems of black and white lattices

In this section we shall apply the decomposition given in the preceding section to derive the factor systems of the three-dimensional black and white lattices. Let T_0 be a three-dimensional lattice and let a_1, a_2 and a_3 be basis translations for T_0. Suppose T is a black and white lattice for which T_0 is the ordinary sublattice. T is then determined by T_0 and an extra translation a with the property $2a \in T_0$.

The inequivalent factor systems of T_0 are obtained by Backhouse (1970) and are given by

$$\gamma(t, t') = \exp(-2\pi i t^{T} A t') \tag{4.1}$$

where t stands for the translation $t = t_1 a_1 + t_2 a_2 + t_3 a_3$ and for the column vector with entries t_1, t_2 and t_3 at the same time, t^{T} is the transpose of t and A is a matrix of the form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ r & s & 0 \end{pmatrix} \tag{4.2}$$

where $p, r, s \in [0, 1)$.

The quotient group T/T_0 equals C_2, consisting of the elements ϵ and α, with $\alpha^2 = \epsilon$. We choose $r(\alpha) = a$, so $m(\alpha, \alpha) = 2a$. For each factor system γ of T_0 we have to solve $P(\alpha, t)$ and $v(\alpha, \alpha)$ from equations (3.4), (3.5) and (3.6) which now take the form

$$P(\alpha, t + t') = P(\alpha, t)P(\alpha, t')\frac{\gamma(t, t')}{\gamma^{*}(t, t')} \qquad \forall t, t' \in T_0 \tag{4.3}$$

$$\gamma(t, 2a) = \gamma(2a, t) \qquad \forall t \in T_0 \tag{4.4}$$

$$\frac{v(\alpha, \alpha)}{v^{*}(\alpha, \alpha)} = P(\alpha, 2a) \tag{4.5}$$

From equation (4.3) it follows that γ^2 must be equivalent to the trivial factor system of T_0. This restricts the possible values of p, r and s to 0 and $\frac{1}{2}$. From these eight

possibilities only two satisfy equation (4.4). One of them is of course the trivial factor system ($p = r = s = 0$) and the other is given in table 1.

Table 1.

$2a$	p	r	s
a_1	0	0	$\frac{1}{2}$
a_2	0	$\frac{1}{2}$	0
a_3	$\frac{1}{2}$	0	0
$a_1 + a_2$	0	$\frac{1}{2}$	$\frac{1}{2}$
$a_1 + a_3$	$\frac{1}{2}$	0	$\frac{1}{2}$
$a_2 + a_3$	$\frac{1}{2}$	$\frac{1}{2}$	0
$a_1 + a_2 + a_3$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

For the two remaining factor systems γ of T_0 we have $\gamma(t, t')/\gamma^*(t, t') = 1$, so the solutions of equation (4.3) are just the one-dimensional unitary representations of T_0 which are well known. For each solution $P(\alpha, t)$ of equation (4.3) there are two solutions $\nu(\alpha, \alpha)$ of equation (4.5). The inequivalent factor systems of T can now be derived with theorem 3 in a straightforward manner. It turns out that there are only two inequivalent factor systems of T. A nontrivial one is given by the nontrivial γ and $P(\alpha, t) = \nu(\alpha, \alpha) = 1$.

5. Factor systems of a class of magnetic space groups

As a second application let us consider a magnetic space group G which is the semi-direct product of a point group P and the invariant subgroup T of translations, which forms a black and white lattice, and let the nonmagnetic subgroup of G also be a semi-direct product. The point group P acts in the natural way on T and on the ordinary sublattice T_0 and leaves both T and T_0 invariant. Of the 517 magnetic space groups which are based on black and white lattices, 110 are of this type.

The problem now is to solve the equations (3.2)–(3.6) where H is the black and white lattice T and K is the point group P. The elements of P may now be chosen to be the coset representatives of G with respect to H. Equation (3.6) now tells us that ν is a factor system of P, and according to theorem 3 we can take this factor system to be real, since every factor system of a point group is equivalent to a real factor system. Therefore ν cancels from equation (3.5) and we only have to solve $P(R, t)$ from the equations

$$P(R, t + t') = P(R, t_1)P^{t_1}(R, t_2)\frac{\gamma(Rt_1, Rt_2)}{\gamma(t_1, t_2)} \qquad \forall t_1, t_2 \in T, \forall R \in P \tag{5.1}$$

and

$$P(R_1 R_2, t) = P(R_1, R_2 t)P(R_2, t) \qquad \forall t \in T, \forall R_1, R_2 \in P. \tag{5.2}$$

The general solution $P_0(R, t)$ of equations (5.1) and (5.2) can be written as

$$P_0(R, t) = P_1(R, t)P(R, t) \tag{5.3}$$

where $P_1(R, t)$ is a particular solution of equations (5.1) and (5.2) and $P(R, t)$ is the general solution of (5.2) and the equation

$$P(R, t_1 + t_2) = P(R, t_1)P^{t_1}(R, t_2) \qquad \forall t_1, t_2 \in T, \forall R \in P. \qquad (5.4)$$

The solutions of equation (5.4) are just the one-dimensional UA representations of T. The reciprocal lattice T_0^* of T_0 is spanned by b_1, b_2 and b_3 where

$$b_i = \frac{2\pi a_j \otimes a_k}{a_i \cdot (a_j \otimes a_k)} \qquad (5.5)$$

where $j = (i+1) \bmod (3)$ and $k = (i+2) \bmod (3)$. As is well known the one-dimensional unitary representations of T_3 are given by $\Delta_k(t) = e^{ik.t}$ where k ranges through the first Brillouin zone of the reciprocal lattice:

$$k = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 \qquad \text{where } \alpha_i \in [0, 1).$$

After a short calculation we find that the one-dimensional UA representations of T are labelled by a pair (k, z) where $z \in U(1)$ and $k \in \Lambda(a)$. $\Lambda(a)$ denotes the set consisting of those four vectors k from the first Brillouin zone that have the properties that $2k$ is a reciprocal lattice vector and $e^{ik.2a} = 1$. For instance if $2a = a_1$ then

$$\Lambda(a) = \left\{ 0, \frac{b_2}{2}, \frac{b_3}{2}, \frac{b_2}{2} + \frac{b_3}{2} \right\}.$$

The one-dimensional UA representations of T are then given by

$$D_{k,z}(t) = e^{ik.t} \qquad D_{k,z}(t+a) = z e^{ik.t} \qquad \forall t \in T_0. \qquad (5.6)$$

The general solution of equation (5.4) is given by $P(R, t) = D_{k(R),z(R)}(t)$. The equations which $k(R)$ and $z(R)$ must satisfy if $P(R, t)$ satisfies equation (5.2) are

$$k(R_1 R_2) = R_2^{-1} k(R_1) + k(R_2) + K(R_1, R_2) \qquad \forall R_1, R_2 \in P \qquad (5.7)$$

and

$$z(R_1 R_2) = z(R_1) z(R_2) e^{ia.[k(R_1 R_2) - k(R_1) - k(R_2) - K(R_1, R_2)]} \qquad \forall R_1, R_2 \in P. \qquad (5.8)$$

Here K is a reciprocal lattice vector which depends on R_1 and R_2. Equation (5.7) is analogous to the equation derived by Backhouse (1970) for the nonmagnetic symmorphic space groups. In that case however k ranges through the whole first Brillouin zone, while here k is restricted to the four vectors in $\Lambda(a)$, which makes equation (5.7) rather easy to solve. Only those solutions of equation (5.7) for which $\sigma(R_1, R_2) = \exp(ia \cdot K(R_1, R_2))$ is a trivial factor system of P have to be considered. For those solutions $z(R)$ must form a projective representation of P with factor system $\sigma'(R_1, R_2) = \exp\{ia \cdot [K(R_1, R_2) - k(R_1, R_2) + k(R_1) + k(R_2)]\}$. This factor system is equivalent with σ. Note that $k(R) \cdot a = 0$ if $2a$ is equal to a_1, a_2 or a_3.

Let us now turn our attention to the problem of finding a solution to equations (5.1) and (5.2). If γ is the trivial factor system of T then of course $P_1(R, t) = 1$ solves equations (5.1) and (5.2). Therefore consider the case where γ is the nontrivial factor system of T.

The factor system γ^R of T defined by $\gamma^R(t, t') = \gamma(Rt, Rt')\gamma^{-1}(t, t')$ is equivalent with the trivial factor system for each $R \in P$, since T has only two inequivalent factor systems. Therefore we can find a mapping $c_R : T \to U(1)$ with $c_R(t_0) = 1$ (t_0 is the unit

element of T) and

$$\gamma^R(t, t') = c_R(t)c_R^t(t')/c_R(t+t').$$

c_R is determined up to multiplication by a one-dimensional UA representation of T. Moreover $c_{R_1}(R_2 t)c_{R_2}(t)c_{R_1 R_2}^{-1}(t)$ is a UA representation of T. Now it is easy to see that equations (5.1) and (5.2) have solutions if and only if the c_R can be chosen such that $c_{R_1}(R_2 t)c_{R_2}(t)c_{R_1 R_2}^{-1}(t) = 1$ for all $t \in T$ and for all $R_1, R_2 \in P$. A solution is then given by $P_1(R, t) = c_R^{-1}(t)$.

Finally we have to apply theorem 3 in order to obtain the inequivalent factor systems. The factor systems obtained so far are given by (γ, ν, P) where γ is one of the two inequivalent factor systems of T, ν belongs to a complete set of inequivalent real factor systems of P and P is a solution of equations (5.1) and (5.2). Let ω and ω' be the factor systems of G given by (γ, ν, P) and (γ', ν', P') respectively. ω and ω' can be equivalent only if $\gamma = \gamma'$ and $\nu = \nu'$. If this is the case then

$$P(R, t)P'^{-1}(R, t) = D_{k_0(R), z_0(R)}(t)$$

for some $k_0 \in \Lambda(a)$ and $z_0 \in U(1)$. ω and ω' are now equivalent if and only if there exists a k_1 in $\Lambda(a)$ and a one-dimensional unitary representation d of P such that

$$k_0(R) = k_1 - R^{-1}k_1 + K \qquad \forall R \in P \tag{5.9}$$

and

$$z_0(R) = d^2(R) \, e^{iRk_0 \cdot a} \qquad \forall R \in P. \tag{5.10}$$

This shows how to obtain a complete set of inequivalent factor systems of G.

Acknowledgment

References

Backhouse N B 1970 *Q. J. Math.* **21** 277–95
Backhouse N B and Bradley C J 1972 *Q. J. Math.* **23** 225–38
Bradley C J and Wallis D E 1974 *Q. J. Math.* **25** 85–99
Mackey G W 1958 *Acta Math.* **99** 265–311